BLOOMINGTON GEOMETRY PROBLEM LIST

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This list is edited by Chris Judge and Bruce Solomon. Please share any comments, suggestions, progress, or relevant information with us.

Problems:

(1) By a *space-time* we mean a four-dimensional manifold X supporting a Lorentzian metric g that satisfies the Einstein equations $R_{ij} - (1/2)Rg_{ij} + \Lambda g_{ij} = 8\pi T_{ij}$. (I.e. "Einstein curvature tensor = matter-energy-stress tensor".) Relative to a diagonalizing basis, the components of T_{ij} represent the energy density ρ , and the three principal pressures p_i. The dominant energy condition requires $\rho \geq p_i$ for each i.

Let X be a 4-manifold homeomorphic to the cone on a compact homology 3-sphere M. (The case $M = \mathbb{S}^3$ underlies the standard closed Robertson-Walker universe with energy density $\Omega > 1$.) Is there any Lorentzian metric on X , suitably spatially symmetric, which satisfies the dominant energy condition? If not, where does the topological obstruction lie?
(Lawrence Brenton)

(2) Consider a smooth hypersurface $X \subset \mathbb{C}P^{n+1}$ of degree $n + 2$, so that the canonical bundle is trivial. Suppose X is defined over $\mathbb{R}$, and has a non-empty real locus $L \subset \mathbb{R}P^{n+1}$. Then X carries a conjugation "invariant" Calabi-Yau ("C-Y") structure:

conjugation reverses the Kähler form, and takes the holomorphic volume form to its conjugate. In this case, L is special Lagrangian with respect to this C-Y structure and (obviously) real-algebraic. Are "nearby" special Lagrangian submanifolds also real-algebraic? This is true for $n = 1$ and $n = 2$ (suitably interpreted), but I don't know it for any $n > 2$.
(Robert Bryant)

(3) Let M be a compact C-Y manifold and let λ be a class in $H_n(M, \mathbb{Z})$ representable by a special Lagrangian ("sLag") submanifold L. By McLean's theorem [McL98], the (compact) moduli space, sLag(λ), of special Lagrangian integral currents that represent λ is smooth near L. In the cases where I can compute sLag(λ), it is a smooth, orientable compact orbifold—even when it contains points that are not smooth sLag submanifolds. Is sLag(λ) always smooth? Orientable? This would be important in the $n = 3$ case for the SYZ conjecture, although observations of Joyce suggest that smoothness may not hold. (see [Jyc])
(Robert Bryant)

(4) (After Misha Brin) Can ergodicity of the *frame flow* fail when M is negatively curved and has full holonomy? The frame flow Φ_t on the bundle FM of positively oriented orthonormal frames on a compact Riemannian manifold M parallel-translates each orthonormal frame $E = \{E_i\}$ in T_pM by t units along the geodesic ray determined by E_1. The metric induces a Φ_t invariant measure on FM. The frame flow is known to be ergodic on an open dense subset of the negatively curved metrics on any M. When the holonomy of M lies in a proper Lie subgroup $G \subset O(n)$, however, ergodicity fails because each subset of $O(n)/G$ generates an invariant set (this occurs for Kähler manifolds of dimension 4 or more, complex and quaternionic projective spaces and the Cayley projective plane).
(Keith Burns)

(5) (After Misha Brin) Is the frame flow (see above) always ergodic on odd-dimensional manifolds? Must it be ergodic if the curvatures are strictly quarter pinched? Since manifolds lacking full holonomy all have even dimension and (after normalization) sectional curvatures ranging from -4 to -1, answers to either of these questions would partially settle the previous one.

The first was answered affirmatively [BG80] in all odd dimensions except 7. Ergodicity is also proven under a very stringent pinching condition in [BG80] and [BP01].
(Keith Burns)

(6) If two compact manifolds M and N are homotopy equivalent, does $\mathrm{MinVol(M^n)} = 0$ imply that $\mathrm{MinVol(N^n)} = 0$? (Recall that MinVol(M) denotes the infimum of the volume of (M, g) over Riemannian metrics g with sectional curvatures lying between -1 and 1.)
(Jianguo Cao)

(7) Let M^n and N^n be positively curved, compact graph-manifolds that admit a homotopy equivalence. (See [Wld67] or [Sch86] for the notion of graph-manifold.) Can M and N have the same marked length-spectrum (or conjugate geodesic flows) without being isometric?
(Jianguo Cao)

(8) If a compact Riemannian manifold has Gromov-hyperbolic universal cover X, can its Gromov norm vanish? Gromov hyperbolicity means that for some constant $C = C(X)$, every loop in X bounds a surface of area $A \leq C\cdot$ (length of the loop).
(Jianguo Cao)

(9) (After Tom Farrell) Suppose a closed manifold M has negative curvature, fundamental group G, and, for some prime p, a monomorphism $Z/p \to \mathrm{Aut}(G)$. Consider the fixed set S of the induced $\mathbb{Z}_p$-action on $\mathbb{S}(\widetilde{M})$ (the sphere at infinity of the universal cover). Does S have finitely generated homology? Is it an absolute neighborhood retract?

Positive answers would bear upon the Nielson realization question. See [FJ88] for partial results.
(Jim Davis)

(10) Let M be a closed, nonpositively curved, locally symmetric Riemannian manifold with no direct factor locally isometric to $\mathbb{R}$. Prove that M has positive Gromov norm. Savage [Sav82] did this for M locally isometric to $\mathrm{SL(n, \mathbb{R})/SO(n, \mathbb{R})}$.
(Benson Farb)

(11) Let M be a closed, nonpositively curved n-manifold with negative Ricci curvature. Bound the degree of any continuous map

$f : N \to M$ for any closed n-manifold N, as follows:

$$\deg(f) \ \le\ C \cdot \frac{\mathrm{Vol(N)}}{\mathrm{Vol(M)}} \ ,$$

with C depending only on n the Ricci curvatures of M and N. Taking $f =$ identity map, such a bound implies MinVol(M) > 0. Connell and Farb [CF] obtain a uniform bound for M locally symmetric with no H^2 and no SL_3/SO_3 factors.
(Benson Farb)

(12) Let $f : M_1 \to M_2$ be a homotopy equivalence between closed 3-manifolds, with M_1 irreducible and M_2 hyperbolic. Does the 'barycenter' mapping [BCG96] derived from f produce a homeomorphism (i.e. explicitly solve the 'homotopy hyperbolic' conjecture)? What if we assume M_1 is hyperbolic?
(Benson Farb)

(13) Call $M^n \subset \mathbb{R}^{n+2}$ *skew* if no two tangent planes to M^n are parallel in $\mathbb{R}^{n+2}$. Skew loops ($n = 1$) can be drawn on any non-quadric closed C^2 surface in $\mathbb{R}^3$. [GS]. Which *negatively* curved surfaces in $\mathbb{R}^3$ contain a skew loop?
(Bruce Solomon)

(14) Give an example of a (compact) skew n-manifold in $\mathbb{R}^{n+2}$ for $n > 1$. An easy argument involving the intersection form on the Grassmannian $G_{2,4}$ rules out any skew surface in $\mathbb{R}^4$ except possibly tori, and Tabachnikov [Tch] shows that no quadric hypersurface $Q^{n+1} \subset \mathbb{R}^{n+2}$ contains a skew n-manifold.
(Bruce Solomon)

(15) (After S. Tabachnikov) Does every non-exact 1-form on $M = \mathbb{S}^3$ differ from a non-vanishing 1-form by an exact 1-form? How about on general M having Euler number zero? Jim Davis points out that by Tischler [Tsh70], when $H^1(M, \mathbb{Z}) \neq 0$, the answer cannot be "yes" unless M is a bundle over $\mathbb{S}^1$.
(Bruce Solomon)

(16) The *parallel overcrossing number* of a knot K is defined to be the minimum, over all isotopic parallels K', of the minimum number of crossings of K' over K in any diagram for the link $K' \cup K$. Does parallel overcrossing number equal crossing number for all knots? I conjecture that it does. It is known to be bounded below by the bridge number of K, and by $2g - 1$ if g is the

genus of K. See [CKS02].
(John Sullivan)

(17) The *ropelength* of a knot is its ratio of length to thickness (the latter being the radius of the largest embedded normal tube around it). Find an explicit non-trivial knot minimal (or even just critical) for ropelength. Plane circles minimize among unknots, and minimizers exist for every knot type [CKS02], but we have no other examples.
(John Sullivan)

(18) The n^{th} *hull* of a knot K is the set of points in space through which every plane cuts K at least $2n$ times. Every nontrivial knot has a nonempty second hull. Is there a topological invariant that guarantees its third hull is nonempty? See [CKKS].
(John Sullivan)

(19) Does the standard triple bubble in $\mathbb{R}^3$ use least area to enclose and separate any three given volumes? I conjecture that it does. See [SM96], and note that the two-dimensional triple bubble problem has been completely solved by Wichiramala [Wch].
(John Sullivan)

(20) Garsia proved that every conformal structure on an orientable surface is realized by an embedded surface in $\mathbb{R}^3$ [Grs62]. His proof, however, is not constructive. Can one find a constructive proof? For tori, Pinkall has done so [Pnk85]: He observed that each conformal torus can be realized by a flat (Hopf) torus embedded in S^3, hence in $\mathbb{R}^3$ by stereographic projection.
(Matthias Weber)

(21) Is there a dodecahedron in $\mathbb{R}^3$ made from twelve congruent, nonregular, strictly convex planar pentagons? (Bending or folding the pentagons not allowed.) The strict convexity of each polygon excludes the example one gets from the rhombic dodecahedron by introducing an artificial vertex on an appropriate side of each rhombus.
(Matthias Weber)

(22) (after M. Herman) Is there any closed Riemannian manifold N whose geodesic flow is stably ergodic but not Anosov? For a discussion of stable ergodicity, see the survey paper [BPSW01].
(Amie Wilkinson)

(23) The time-1 map of an Anosov flow is always partially hyperbolic (see [BPSW01] for a survey of partial hyperbolicity). Is there any Riemannian manifold N whose geodesic flow $\varphi : SN \to SN$ is *not* Anosov, but whose time-1 map φ_1 is partially hyperbolic? **(Amie Wilkinson)**

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